Whitehead's theorem: homotopy groups of CW-complexes carry a lot of information!

Thm: $f: X \to Y$ map b/w connected CW-complexes, $f_*: \pi_n(X) \to \pi_n(Y)$ isom. $\forall n$
$\Rightarrow$ f is a homotopy equivalence
(ie. $\exists g: Y \to X$ st. $fg \simeq id$ and $gf \simeq id$).

If f is inclusion of a subcomplex then stronger statement holds:
X is a deformation retract of Y.

- **Rmk:** Whitehead's thm does not say that two spaces with isomorphic π_n are homotopy equivalent! The isoms. have to be induced by a map f.

  Ex: $X = \mathbb{RP}^2 \times S^\infty \; (\simeq \mathbb{RP}^2)$ $\qquad$ $Y = S^2 \times \mathbb{RP}^\infty$

  $\pi_1(X) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$ $\qquad$ $\pi_1(Y) = \pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$

  $\tilde{X} = \tilde{Y} = S^2 \times S^\infty$ so same π_n $\forall n \geq 2$

  However $H_k(X) = H_k(\mathbb{RP}^2) \neq 0$ only for $k \leq 2$ $\qquad$ (recall S^∞ contractible) ($H_*(S^\infty)$ trivial easy!)

  $H_k(Y) \neq 0$ for ∞ly many k since $\mathbb{RP}^\infty$ has ∞ly many nonzero homology groups.

  so $X \not\simeq Y$!

- **Rmk:** similar statement for H_* is false: take X with $H_*(X) \simeq H_*(pt)$ but $\pi_1(X) \neq 0$ (eg. Poincaré sphere minus a point), $f: X \xrightarrow[\text{const.}]{} pt$

Lecture 4, Mon 1/30

However there's a version of Whitehead's thm in homology assuming X, Y simply conn^d (see later)

Proof:

key technical lemma: Compression lemma:

(for $n=0$: $\pi_0 B \twoheadrightarrow \pi_0 X$)

(X, A) CW-pair, (Y, B) any pair with $B \neq \emptyset$.
$\forall n$ st. $X - A$ has n-dim. cells, assume $\pi_n(Y, B, y_0) = 0$ $\forall y_0 \in B$.
Then every map $f: (X, A) \to (Y, B)$ is homotopic rel. A to a map $X \to B$.

Pf: by induction on dim., assume f has been homotoped so that it maps the $(k-1)$-skeleton X^{k-1} to B. Consider a k-cell e^k of $X - A$, with characteristic map $\phi: D^k \to X$, then $f \circ \phi: (D^k, \partial D^k) \to (Y, B)$.

Since $\pi_k(Y,B,y_0)=0$, can homotope $f\circ\phi$ rel. ∂D^k so it maps into B
(by compression criterion)

$\Rightarrow$ induces homotopy of f on $X^{k-1}\cup e^k$, rel. X^{k-1}.

Doing this on all k-cells, get a homotopy of $f_{|X^k\cup A}$ to a map into B.
(& not moving A)

By homotopy extension property for CW-pairs, can extend this homotopy to all X.

Proceed by induction. (if dim ∞, do steps in time $\frac{1}{2^k}$ so it converges to a homotopy $[0,1]\times X\to Y$, well behaved since any cell eventually stationary). ∎

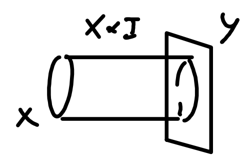
Pf. of Whitehead's thm:

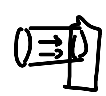
1) special case where $f: X\to Y$ inclusion of subcomplex: consider long exact seq. in relative homotopy:
$$\cdots\to\pi_n(X)\xrightarrow[f_*]{\cong}\pi_n(Y)\to\pi_n(Y,X)\xrightarrow{\partial}\pi_{n-1}(X)\xrightarrow[f_*]{\cong}\pi_{n-1}(Y)\cdots$$
so $\pi_n(Y,X)=0\ \forall n$.

Then by compression lemma, the identity map $id:(Y,X)\to(Y,X)$ is homotopic rel. X to a map $r: Y\to X$, $r_{|X}=id_X$
ie. a deformation retraction of Y onto X (in particular a homotopy equiv^ce^).

2) general case: consider mapping cylinder of f.

$$M_f=(X\times I)\sqcup Y\Big/(x,1)\sim f(x)$$



M_f contains both $X\times\{0\}\simeq X$ and Y as subspaces and retracts onto Y ()

f = composition of inclusion $X\overset{i}{\hookrightarrow}M_f$ and retraction $M_f\to Y$
$\simeq$ homotopy eq^ce^.

$\Rightarrow$ enough to show $X\overset{i}{\hookrightarrow}M_f$ is a homotopy equiv^ce^.

Note: since f_* isoms on all π_n, so is i_*.

IF f is a cellular map taking n-skeleton $X^n\to Y^n\ \forall n$, then M_f is clearly a cell complex and (M_f,X) is a CW-pair so thm follows from above special case.

Otherwise: either use a trick (see Hatcher) or cellular approximation thm:
f is homotopic to a cellular map. (will see soon). ∎

Cellular approximation:

We'd like to prove $\pi_n(S^k)=0$ for $n<k$ by just arguing that a map $S^n\to S^k$ must miss some point $q\in S^k$, then contract $S^k-\{q\}$ to $*$ ...
but first need to ensure the map isn't surjective! (space-filling curves...)

In fact, when studying maps b/w CW-complexes, can reduce to

cellular maps := $f:X\to Y$ st $f(X^n)\subseteq Y^n$ $\forall n$ (maps n-cells to cells of dim. $\leq n$).

Thm (cellular approximation):

Every map $f:X\to Y$ of CW-complexes is homotopic to a cellular map.
If f already cellular on subcomplex $A\subset X$, can take homotopy stationary on A.

Corollary: $\pi_n(S^k)=0$ for $n<k$.

(pf: S^k = 0-cell $\cup$ k-cell, S^n = 0 cell $\cup$ n-cell. Any map $S^n\to S^k$ can be homotoped to a cellular map, i.e. constant map).

This is similar to simplicial approximation thm for simplicial complexes (simplicial maps are cellular!) but doesn't require subdivision of the domain.

Pf of thm: By induction on dim. Assume f cellular on $(n-1)$-skeleton X^{n-1}, let e^n be an n-cell of X. Since $\overline{e^n}\subset X$ compact, $f(\overline{e^n})$ compact hence intersects finitely many cells of Y. [weak topology: seq. of pts in ∞ diff cells wouldn't converge in Y!]

Let $e^k\subset Y$ cell of highest dim. meeting $f(e^n)$; if $k\leq n$ $f_{|e^n}$ already cellular ✓

So assume $k>n$; we'll show:

Claim: can homotope $f_{|e^n}: e^n\to Y^k$ rel. boundary & only in e^k so that it misses some point $p\in e^k$

Then get $f(e^n)$ to miss all of e^k by composing with a retraction of Y^k-p to Y^k-e^k.
Repeat (finitely many times) to eventually get $f(e^n)$ to miss all cells of dim $>n$.

Do this for all n-cells $\Rightarrow$ get homotopy of $f_{|X^n}$ rel $X^{n-1}\cup A$ to cellular map.

By homotopy extension property for CW-pairs, can extend this homotopy to all X.

Proceed by induction. (if dim ∞, do steps in time $\frac{1}{2^k}$ so it converges to a homotopy $[0,1]\times X\to Y$, well def^d since any cell eventually stationary).

Pf Claim: follows from a linear approximation lemma.

Def: • polyhedron in $\mathbb{R}^n$:= union of finitely many convex polyhedra

= compact set, $\cap$ finitely many half spaces delimited by hyperplanes

• PL (piecewise linear) map polyhedron $\to \mathbb{R}^k$:= $\exists$ decomp. into convex polyhedra s.t. linear on each

Lemma: $f: I^n \to Z = W \cup \underset{\text{cell}}{e^k}$, then f is homotopic rel $f^{-1}(W)$ to a map f_1 for which $\exists$ polyhedron $K \subset I^n$ s.t.

(1) $f_1(K) \subset e^k$, $f_1(K)$ is PL wrt $e^k \cong \mathbb{R}^k$

(2) $K \supset f_1^{-1}(U)$ for some open $U \neq \emptyset \subset e^k$.

Apply this to our situation: for $k > n$, we have $f|_{e^n}: e^n \to Y^k = W \cup e^k$, $k > n$.
Lemma gives map f_1 which only differs from f inside e^k and is PL on some $K \subset e^n$; since $f_1|_K$ can't be onto U ($k > n$), f_1 misses a point of e^n ✓

▲

Pf. Lemma: Identify $e^k \cong \mathbb{R}^k$, let $B_1 \subset B_2 \subset e^k$ balls of radii 1 & 2.

$f^{-1}(B_2)$ closed $\subset I^n$, hence compact $\Rightarrow$ f is unif. continuous on $f^{-1}(B_2)$

$\exists \varepsilon$ s.t. $x, y \in f^{-1}(B_2)$, $|x-y| < \varepsilon \Rightarrow |f(x) - f(y)| < \frac{1}{2}$.

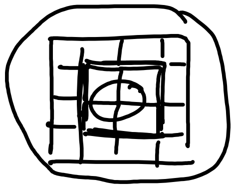
Also can ensure $\varepsilon < \frac{1}{2} \operatorname{dist}(f^{-1}(B_1), I^n - f^{-1}(\operatorname{int} B_2))$.

disjoint compacts

Subdivide I^n into cubes of diameter $< \varepsilon$, $K_1 = \bigcup$ all cubes meeting $f^{-1}(B_1)$

K_2 = add adjacent cubes to those.

$K_2 \subset f^{-1}(B_2)$.



$f^{-1}(B_1) \subset K_1 \subset K_2 \subset f^{-1}(B_2) \subset I^n$

Next subdivide K_2's cubes into simplices (, , inductively)

$g: K_2 \to e^k$ PL map that $= f$ at vertices of K_2 & linear on simplices

$\varphi: K_2 \to [0,1]$ cut off function, $\varphi|_{K_1} = 1$, $\varphi|_{\partial K_2} = 0$, then set

$f_t = (1 - t\varphi) f + t\varphi g$ on K_2, f elsewhere.

$f_t = f$ outside K_2 hence on $f^{-1}(W)$; f_1 is PL on K_1; $\sup|f_1 - f| < \frac{1}{2}$ so $f_1^{-1}(B_{1/2}) \subset K_1$ ▲

Remark: can also use cellular approx. for maps of pairs!

Every map $f:(X,A)\to(Y,B)$ can be deformed through maps of pairs to a cellular map.

Indeed, first deform $f|_A : A \to B$ to be cellular; then extend this to a homotopy of f on all of X; then deform resulting map to a cellular one, remaining fixed on A.

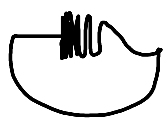
Corollary: given a CW-pair (X,A),
If all cells of $X-A$ have dim. $>n$ then (X,A) is n-connected.
In particular (X,X^n) is n-connected. Hence $X^n \hookrightarrow X$ induces isoms on π_k, $k<n$ and surjection on π_n.

Pf.: given a map $(D^k,\partial D^k)\to(X,A)$, where $k\leq n$, cellular approx. deforms it to a cellular map, which sends D^k to A.
This proves $\pi_n(X,A)=0$ (every map $(D^k,\partial D^k)\to(X,A)$ deforms into A).
The second statement follows from long exact seq. for relative π_k. ▲

CW-approximation:

Def: $f: X\to Y$ is a <u>weak homotopy eq^ce</u> if induces isoms. f_* on $\pi_n(X,x_0)$ $\forall n$ and $\forall$ base point.

By Whitehead, a weak homotopy eq. between CW-complexes is a homotopy eq.
Hence also true for spaces which are homotopy eq. to CW-complexes!

* However, $\exists$ spaces which have all homotopy groups trivial, but aren't contractible, eg "quasicircle"  ; these are weakly homotopy eq. to a point, but not homotopy eq. to it (or to any CW-complex)

so: in general weakly h.e. $\Leftarrow$ h.e.
$\not\Rightarrow$

Def: A CW-approximation of X is a CW-complex Z + a weak h.e. $f: Z\to X$.

[Prop: a weak h.e. induces isoms. on all homotopy, homology & cohom. groups; (will follow from Hurewicz) so for many arguments we can reduce to the case of CW-complexes!]

May want variants on this theme, eg. – CW-pair approx. (Z,A) of pair (X,A) ($A\subset X$ CW-complex)
– approx. up to dim. n / in dims $>n$ only.

<u>Construction of CW-approx. of (X,A)</u>: $A \subset X$, A CW-complex, $\pi_0(A) \twoheadrightarrow \pi_0(X)$
(eg. 1 point in each component of X)

By induction, build Z as union of subcomplexes $A = Z_0 \subset Z_1 \subset \dots$
where
- Z_k obtained from Z_{k-1} by attaching k-cells
- $f: Z_k \to X$ identity on A and f_* isom. on π_i, $0 \le i < k$
  surj. on π_k

- given Z_k: choose cellular maps $\varphi_\alpha : S^k \to Z_k$ representing generators for the kernel of $f_* : \pi_k(Z_k) \to \pi_k(X)$ (for all conn. components fixing a basept in each).

  Attach cells e_α^{k+1} to Z_k via φ_α, call resulting CW-complex Y_{k+1}.
  Since $f \circ \varphi_\alpha$ nullhomotopic, can extend f over e_α^{k+1} to get $f: Y_{k+1} \to X$.

  <u>Claim:</u> $f_* : \pi_k(Y_{k+1}) \to \pi_k(X)$ isom.

  <u>Pf:</u>
  - it is injective since elts of kernel can be represented by cellular maps, hence maps into Z_k, and those are nullhomotopic in Y_{k+1} by construction.
  - it still is surjective, since the surjective map $\pi_k(Z_k) \xrightarrow{f_*} \pi_k(X)$ factors through.
$$\begin{tikzcd} \pi_k(Z_k) \arrow[rr, "f_*"] \arrow[dr] & & \pi_k(X) \\ & \pi_k(Y_{k+1}) \arrow[ur, "f_*"'] & \end{tikzcd}$$
  Δ

  and π_i, $i<k$ are not affected by attaching $(k+1)$-cells (cellular approx.! $\Rightarrow$ representatives of π_i and homotopies live in $(i+1)$-skeleton).

- for $k=0$; Y_1 = attach 1-cells joining all basepts 0-cells of $Z_0 = A$ which lie in same connected component of X. Then clearly $\pi_0(Y_1) \xrightarrow{\simeq} \pi_0(X)$.

- Next, choose $\psi_\beta : S^{k+1} \to X$ generators of $\pi_{k+1}(X)$ for given base pts in each component of X
  $$Z_{k+1} = Y_{k+1} \vee \left(S_\beta^{k+1}\right)_\beta$$
  (wedge sphere S_β^{k+1} at base pt of appropriate connected component)
  and extend f by ψ_β on S_β^{k+1}.

  This guarantees surjectivity of f_* on $\pi_{k+1}(Z_{k+1}) \to \pi_{k+1}(X)$
  Moreover, for $i \le k$ we have that $\pi_i(Y_{k+1}) \underset{\text{incl.}}{\xrightarrow{\simeq}} \pi_i(Z_{k+1})$ ✓
  indeed $\pi_i(Y_{k+1}) \twoheadrightarrow \pi_i(Z_{k+1})$ by cellular approx.
  $\pi_i(Y_{k+1}) \hookrightarrow \pi_i(Z_{k+1})$ because composition to $\pi_i(X)$ is injective.

- Taking $Z = \bigcup_k Z_k$, since maps $\pi_i(Z) \to \pi_i(X)$ only depend on $(i+1)$-skeleton, they're isos. as well.

Δ

Rmk: when (X,A) is n-connected, long exact seq. $\Rightarrow$ $\pi_k(A) \xrightarrow{\sim} \pi_k(X)$ $k<n$
$\pi_n(A) \twoheadrightarrow \pi_n(X)$

so can skip first n steps of construction...

Corollary: (X,A) n-connected $\Rightarrow$ $\exists$ CW-approx. (Z,A) st. all cells of $Z-A$ have dimension $\geq n+1$.

- can also choose to start at step n no matter what, to build a CW-complex that looks "like A up to π_n and like X after π_{n+1}". (see Hatcher Prop 4.13)

Example of similar trick: Postnikov towers

Prop: X CW complex (connected for simplicity) $\Rightarrow$ $\exists$ CW-complex X_n with $\pi_i(X_n) = \begin{cases} \pi_i(X) & i \leq n \\ 0 & i > n \end{cases}$

and these fit into a tower + comm diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_2 \arrow[d] \\
 & X_1 \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & X_0
\end{tikzcd}$$

Namely: • to build X_n, start from X and take $\varphi_\alpha : S^{n+1} \to X$ generating $\pi_{n+1}(X)$

Attach $(n+2)$-cells e^α along φ_α to get a CW-complex Y.

Cellular approx. $\Rightarrow$ $\pi_i(X) \xrightarrow{\sim} \pi_i(Y)$ for $i \leq n$

but $\pi_{n+1}(Y) = 0$: by cellular approx. any $S^{n+1} \to Y$ can be homotoped into X, then by construction it's nullhomotopic in Y.

Then attach $(n+3)$-cells to Y along generators of $\pi_{n+2}(Y)$ to kill it, and so on ... to get X_n.

• The inclusion $X \overset{i}{\hookrightarrow} X_n$ extends to a map $X_{n+1} \to X_n$ since X_{n+1} obtained by attaching cells of dim $\geq n+3$ to X, and $\forall k \geq n+3$, $\pi_{k-1}(X_n) = 0$ $\Rightarrow$ attaching map of k-cell is nullhomotopic in X_n $\Rightarrow$ can extend i over k-cell $\rightsquigarrow$ i extends to X_{n+1}.